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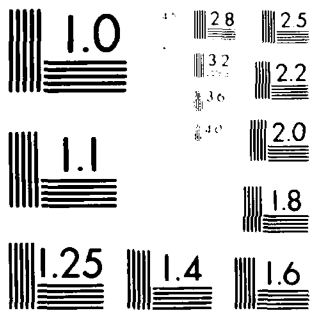

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CONTROLLABILITY and STABILIZABILITY of

REGULAR SINGULAR LINEAR SYSTEMS with CONSTANT COEFFICIENTS+

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CONTROLLABILITY AND STABILIZABILITY OF
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Abstract. A concept of controllability for systems $A\dot{x} + Bx = Cu(t)$ in which A may be singular is introduced. When $\det(As+B) \not\equiv 0$, $s \in \mathbb{C}$, this is shown to be equivalent to the condition that $c^T(As+B)^{-1}C \equiv 0$ implies $c = 0$. It is also shown that when such a system is controllable and C is a column vector, then there exists a feedback $u = g_0^T x + g_1^T \dot{x}$ such that $A - Cg_1^T$ is non-singular and all solutions of $A\dot{x} + Bx = C(g_0^T x + g_1^T \dot{x})$ decay exponentially.

1. Introduction

We consider systems of the form

$$\text{(1.1)} \qquad A\dot{x} + Bx = Cu \qquad (\dot{x} = dx/dt)$$

where x and u are $\mathbb{R}^n$- and $\mathbb{R}^m$-valued functions, respectively, of $t \in \mathbb{R}$ and where the constant matrices A and B are $n \times n$ and C is $n \times m$ and all have real elements. We shall say the system is <u>singular</u> when A is singular and <u>regular</u> when

$$\text{(1.2)} \qquad \Delta(s) = \det(As+B) \not\equiv 0, \qquad s \in \mathbb{R} \text{ (or } \mathbb{C}\text{)}.$$

Condition (1.2) is the criterion for the pencil of matrices $\{As+B\}$ to be regular in the terminology of [2], p.25.

Our results for regular singular systems (1.1) reduce to well-known facts for the case when A is non-singular. Regular singular systems of the form

$$\text{(1.3)} \qquad A\dot{x} + Bx = f(t)$$

have been treated elsewhere [1] where a formula for the solutions of (1.3) is given. We give an alternative development and an equivalent formula in §2. A more detailed exposition is in [3].

It is known and will become evident in §2 that when a solution of (1.1) or (1.3) exists for given initial conditions, then the

solution is unique. When f in (1.3) or u in (1.1) is continuous on an interval $\mathcal{J}$, then a solution x on $\mathcal{J}$ will be understood to be a function which is differentiable on $\mathcal{J}$ and which satisfies the equation everywhere on $\mathcal{J}$. Thus, in treating the concept of controllability in regard to (1.1), we restrict the controls u to the class of continuous $\mathbb{R}^m$-valued functions and require differentiability of the corresponding responses x. We shall see that when control can be effected, the function u can, in fact, be chosen in a class

$$(1.4) \qquad \mathcal{U}_\mu = \{u:[t_0,\infty) \to \mathbb{R}^m : u^{(k)} \text{ is continuous}, \quad k=0,1,\ldots,\mu\}$$

for some $\mu > 0$.

Definition 1.1.

System (1.1) is controllable (at time t_0) if for every $\xi,\zeta \in \mathbb{R}^n$, there is a $u \in \mathcal{U}_0$ and a $\tau > 0$ such that there is a solution x of (1.1) satisfying $x(t_0) = \xi$ and $x(t_0+\tau) = \xi$; it is controllable from zero if the same is true with the restriction $\xi = 0$.

Remark 1.2.

It should be clear that (1.1) is controllable at time t_0 if and only if it is controllable at time 0. Since A and B are constant, one need only translate the control which effects

the transfer from ξ to ζ. Accordingly, unless otherwise noted, we take $t_0 = 0$ in (1.4) and Definition 1.1.

Our main result, proved in §3, is the following.

Theorem 1.3.

Let (1.1) be regular. Then (1.1) is controllable if and only if

(1.5) $\quad c \in \mathbb{R}^n, \quad c^T(As+B)^{-1}C \equiv 0, \quad s \in \mathbb{R}, \quad$ implies $\quad c = 0.$

Moreover, when (1.1) is controllable, we may take $\tau > 0$ arbitrarily, regardless of ξ, ζ.

Remark 1.4.

If the coefficient matrices A,B,C have elements in $\mathbb{C}$, the complex numbers, then our results are valid also in the corresponding context; that is, we may replace $\mathbb{R}^n$ and $\mathbb{R}^m$ in (1.4), Definition 1.1 and (1.5) by $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, with x being $\mathbb{C}^n$-valued. It will be clear that our proofs remain valid in this context when appropriate trivial modifications are made. It should be noted that $(As+B)^{-1}$ is not generally defined for all $s \in \mathbb{R}$ (or $\mathbb{C}$). However, when (1.2) holds, the elements of $(As+B)^{-1}$ are rational functions of s and (1.5) is to be interpreted to mean that the elements of $c^T(As+B)^{-1}C$ are each the zero function at points s where they are defined. Note that it is irrelevant

whether s is interpreted as a variable over $\mathbb{R}$ or over $\mathbb{C}$.

Remark 1.5.

Singularity of A is not an hypothesis in Theorem 1.3. When A is non-singular it is easy to show by expanding $(As+B)^{-1}$ in powers of s^{-1} for large s that (1.5) is equivalent to the condition

$$\operatorname{rank}[B_1, A_1B_1, \ldots, A_1^{n-1}B_1] = n, \quad B_1 = A^{-1}C,\ A_1 = -A^{-1}B.$$

This is the well-known condition (Theorem 5, p.81, [5]) for controllability of the system $\dot{x} = A_1x + B_1u$ equivalent to (1.1) when A^{-1} exists.

In §4 we prove a stabilizability result for (1.1). It is convenient to use the following.

Definition 1.6.

System (1.1) is stabilizable if there exist real $m \times n$ matrices G_0 and G_1 such that with

$$u = G_0x + G_1\dot{x} \tag{1.6}$$

in (1.1) the resulting system

$$(1.7)\qquad \tilde{A}\dot{x} + \tilde{B}x = 0, \quad \tilde{A} = A - CG_1, \quad \tilde{B} = B - CG_0,$$

has $\tilde{A}$ non-singular and all solutions tend to zero exponentially as $t \to +\infty$, or, equivalently, $\det(\tilde{A}s+\tilde{B})$ has degree n and all its zeros have negative real parts.

The main result in §4 is Theorem 4.1 which states that if (1.1) is regular and controllable and $m = 1$, then (1.1) is stabilizable. Corollary 4.2 affirms the analogous result for certain cases with $m > 1$. We intend to treat the general case $m > 1$ at a later time.

2. Decomposition of the system.

Throughout this section we assume A is singular and we sketch briefly an analysis leading to an explicit formula for the solutions of (1.1) in this case. Additional details can be found in [3]. Condition (1.2) implies the existence of a Laurent expansion for $(A+zB)^{-1}$ in a deleted neighborhood of zero; that is,

$$(2.1)\qquad (A+zB)^{-1} = \sum_{k=-\mu}^{\infty} z^k Q_k, \quad z \in \mathbb{C}, \quad 0 < |z| < \delta$$

for some $\delta > 0$. Here $Q_{-\mu} \neq 0$, $\mu \geq 1$ since A is singular and the $n \times n$ matrices Q_k have real elements when A and B do. (Many of the relations which follow appear in [4] but derived from a different point of view. It should be noted that we used

$\mu + 1$ there in place of μ here.)

From (2.1) it readily follows that

$$Q_k A = -Q_{k-1}B, \quad AQ_k = -BQ_{k-1}, \quad k \neq 0 \tag{2.2}$$
$$Q_0 A + Q_{-1}B = I_n \quad , \quad AQ_0 + BQ_{-1} = I_n$$

where I_n denotes the $n \times n$ identity matrix. One may now show that

$$AQ_k B = BQ_k A, \quad k \geq -\mu, \tag{2.3}$$

and that

$$Q_k AQ_j = Q_j AQ_k = \begin{cases} 0 & , \quad k \leq -1, \ j \geq 0 \\ Q_{k+j}, & k \geq 0, \quad j \geq 0 \\ -Q_{k+j}, & k \leq -1, \ j \leq -1. \end{cases}$$

From these, (2.2) and (2.3) it follows that

$$Q_0 AQ_0 = Q_0 \quad , \quad Q_{-1}BQ_{-1} = Q_{-1} \tag{2.4}$$

and since $Q_{-1} \neq 0$, that

$$(Q_{-1}A)^{\mu} = 0 \ , \ (Q_{-1}A)^{\mu-1} \neq 0. \tag{2.5}$$

If we define

$$P_0 = Q_0 A, \quad P_1 = Q_{-1} B, \tag{2.6}$$

then from (2.2) and (2.5) we see that P_0 and P_1 are complementary projections;

$$P_i^2 = P_i, \quad i = 0,1; \quad P_0 + P_1 = I_n. \tag{2.7}$$

If we let $r = \text{rank } P_0$, then $r < n$ since A is assumed to be singular. For convenience we assume $r > 0$; the case $r = 0$ is included in what follows if one omits various terms which are vacuous, in effect, in that case. In a similar way the case $r = n$ (A non-singular) is included in what follows. Accordingly, we define $\rho = n - r$ and consider that

$$r > 0, \quad \rho = n - r > 0. \tag{2.8}$$

Now let X be $n \times r$ and Y be $n \times \rho$ with the columns of X and Y forming bases for the ranges (column spaces) of P_0 and P_1, respectively. By (2.7) the $n \times n$ matrix

$$T = [X, Y] \tag{2.9}$$

is non-singular and we define $r \times n$ and $\rho \times n$ matrices U and V, respectively, by

$$(2.10) \qquad T^{-1} = \begin{bmatrix} U \\ V \end{bmatrix}.$$

The following relations then hold:

$$(2.11) \qquad UX = I_r,\ UY = 0,\ VX = 0,\ VY = I_\rho,\ XU + YV = I_n.$$

Since $P_0X = X$ and $P_1Y = Y$ relations (2.11) imply

$$(2.12) \qquad P_0 = XU, \quad P_1 = YV$$

$$(2.13) \qquad UP_0 = U, \quad UP_1 = 0, \quad VP_0 = 0, \quad VP_1 = V.$$

Below we shall need the $r \times r$ matrix β and $\rho \times \rho$ matrix α defined by

$$(2.14) \qquad \beta = -UQ_0BX \quad , \quad \alpha = -VQ_{-1}AY.$$

Using (2.3) and (2.4), one finds that $Y\alpha V = -Q_{-1}A$ and (2.5) then implies

$$(2.15) \qquad \alpha^\mu = 0 \quad , \quad \alpha^{\mu-1} \neq 0.$$

We shall also need the $n \times n$ matrix

$$S = \begin{bmatrix} UQ_0 \\ VQ_{-1} \end{bmatrix} . \tag{2.16}$$

Observe that $[AX,BY]S = AP_0Q_0 + BP_1Q_{-1} = I_n$ by (2.12), (2.4) and (2.2). Hence

$$S^{-1} = [AX,BY]. \tag{2.17}$$

Lemma 2.1.

Let (1.1) be regular and suppose (2.8) holds. Then (1.1) is equivalent to

$$\dot{x}_0 - \beta x_0 = \Gamma_0 u \tag{2.18_0}$$

$$x_1 - \alpha \dot{x}_1 = \Gamma_1 u \tag{2.18_1}$$

where

$$x = T \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = Xx_0 + Yx_1, \quad x_0 \in \mathbb{R}^r, \quad x_1 \in \mathbb{R}^\rho \tag{2.19}$$

and

$$\Gamma_0 = UQ_0C \quad , \quad \Gamma_1 = VQ_{-1}C. \tag{2.20}$$

Proof:

Let

$$\tilde{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = T^{-1}x \tag{2.21}$$

so that

$$x_0 = Ux, \quad x_1 = Vx. \tag{2.22}$$

Using the substitution (2.19), we see that (1.1) is equivalent to

$$(SAT)\dot{\tilde{x}} + (SBT)\tilde{x} = SCu. \tag{2.23}$$

Computing the submatrices comprising SAT we find

$$UQ_0AX = I_r, \quad UQ_0AY = 0, \quad VQ_{-1}AX = 0, \quad VQ_{-1}AY = -\alpha$$

by virtue of (2.3), (2.11), (2.12), (2.13) and (2.14). Similar computations give the submatrices comprising SBT and we have

$$SAT = \begin{bmatrix} I_r & 0 \\ 0 & -\alpha \end{bmatrix}, \quad SBT = \begin{bmatrix} -\beta & 0 \\ 0 & I_\rho \end{bmatrix}. \tag{2.24}$$

It is now evident that (1.1) is equivalent to (2.18_i), $i = 0,1$, by virtue of (2.21) and (2.20). [#]

Lemma 2.2.

Let x_1 be a solution of (2.18_1) on an interval $\mathcal{J}$ with u continuous on $\mathcal{J}$. If we define

$$(2.25) \qquad w_k = \alpha^{\mu-k} x_1, \qquad k = 0,1,\ldots,\mu,$$

then $\dot{w}_k$ exists on $\mathcal{J}$ and

$$(2.26) \qquad w_{k+1} = \alpha^{\mu-k-1} \Gamma_1 u + \dot{w}_k, \qquad k = 0,\ldots,\mu-1.$$

Each w_k, $k = 0,1,\ldots,\mu$ is uniquely determined by u and if u is $j-1$ times differentiable on $\mathcal{J}$, then

$$(2.27) \qquad w_j = \sum_{k=0}^{j-1} \alpha^{\mu-j+k} u^{(k)}, \qquad j = 1,\ldots,\mu.$$

Proof:

By (2.25) $\dot{w}_k = \alpha^{\mu-k}\dot{x}_1$ exists on $\mathcal{J}$ and by (2.18_1) we have, for $0 \le k \le \mu - 1$,

$$w_{k+1} = \alpha^{\mu-k-1}(\alpha\dot{x}_1 + \Gamma_1 u)$$

which, in effect, is (2.26). Since $w_0 = \alpha^{\mu} x_1 = 0$ by (2.15), we see that w_0 is uniquely determined by u. The same follows inductively for $w_1,\ldots,w_\mu$ by (2.26). With $k = 0$ in (2.26) we have $w_1 = \alpha^{\mu-1}\Gamma_1 u$ which is (2.27) for $j = 1$. An inductive proof using (2.26) then establishes (2.27) for any j, $1 \leq j \leq \mu$, when u is $j - 1$ times differentiable. [#]

Remark 2.3.

Taking $k = \mu$ in (2.25), we see that the previous lemma implies that $x_1 = w_\mu$ is a formula for the solution of (2.18_1) when u is continuous. For example, if $\mu = 3$ one gets from (2.26) and the fact that $w_0 = 0$, that

$$x_1 = w_3 = \Gamma_1 u + \frac{d}{dt}(\alpha\Gamma_1 u + \frac{d}{dt}\alpha^2\Gamma_1 u)$$

with no implication that $\dot{u}$ and $\ddot{u}$ exist. When u is $\mu - 1$ times differentiable on $\mathscr{I}$, then (2.27) with $j = \mu$ gives the easily written formula

$$(2.28) \qquad x_1(t) = \sum_{k=0}^{\mu-1} \alpha^k \Gamma_1 u^{(k)}(t), \qquad t \in I.$$

In any case, with u continuous on $\mathscr{I}$ the solution of (2.18_0) with $x_0(0)$ given is

$$(2.29) \qquad x_0(t) = e^{\beta t}x_0(0) + \int_0^t e^{-\beta(t-s)}\Gamma_0 u(s)ds.$$

Observe that whereas $x_0(0)$ can be chosen arbitrarily and independently of u, the value of $x_1(0)$ is determined by u. Formulas (2.28) and (2.29) are equivalent to equation (19), p.419, in [1].

3. Controllability of the system.

Controllability for (1.1) clearly implies that (2.18_0) and (2.18_1) are each controllable. Whereas criteria for controllability of (2.18_0) are well-known, this seems not to be the case for (2.18_1). In connection with the latter the following is important.

Lemma 3.1. Let x_1 be a solution of (2.18_1) on an interval $\mathcal{J}$ where u is continuous. If $c_1 \in \mathbb{R}^\rho$ satisfies

$$(3.1) \qquad c_1^T \alpha^k \Gamma_1 = 0, \quad k = 0,\dots,\mu-1,$$

then

$$(3.2) \qquad c_1^T x_1(t) \equiv 0 \qquad \text{on } \mathcal{J}.$$

Proof:

When u is $\mu - 1$ times differentiable, this follows immediately from (2.28). If u is merely continuous, then (3.2) follows from (3.1) by an induction argument employing (2.26) in-as much as $w_0 = 0$ and $w_\mu = x_1$. #

Theorem 3.2.

Let (1.1) be regular. In order that it be controllable it is necessary that

$$\text{(3.3)} \qquad \operatorname{rank} H_0 = r, \quad \operatorname{rank} H_1 = \rho$$

where

$$\text{(3.4)} \quad H_0 = [\Gamma_0, \beta\Gamma_0, \ldots, \beta^{r-1}\Gamma_0], \quad H_1 = [\Gamma_1, \alpha\Gamma_1, \ldots, \alpha^{\mu-1}\Gamma_1].$$

Proof:

It was pointed out above that if (1.1) is controllable, then both (2.18_0) and (2.18_1) are controllable. We must then have $\operatorname{rank} H_0 = r$. The condition $\operatorname{rank} H_1 = \rho$ follows from Lemma 3.1. Indeed if $\operatorname{rank} H_1 < \rho$ then there is a $c_1 \in \mathbb{R}^r$, $c_1 \neq 0$ such that $c_1 H_1 = 0$. But then by (3.2) we have $c_1^T x_1(\tau) = 0$ so there can be no control $u \in \mathscr{U}_0$ transferring x from $x(0) = 0$ to $x(\tau) = Yc_1$ for $\tau > 0$ inasmuch as then $x_1(\tau) = c_1$ (cf. (2.19) and (2.21)) for which $c_1^T x_1(\tau) \neq 0$. #

Theorem 3.3.

Let (1.1) be regular. Then it is controllable from zero if and only if (3.3) holds. Moreover, when (3.3) holds, then $\tau > 0$ in Definition 1.1 can be chosen arbitrarily and one can choose $u \in \mathscr{U}_\mu(t_0=0)$ such that

$$(3.5) \qquad u(0) = u'(0) = \cdots = u^{(\mu)}(0) = 0$$

so that

$$(3.6) \qquad \dot{x}(0) = 0 \quad \text{when} \quad x(0) = 0.$$

Proof:

The proof given for Theorem 3.2 suffices to show that (3.3) is necessary for (1.1) to be controllable from zero. To prove that (3.3) is sufficient we restrict ourselves to controls $u \in \mathscr{U}_\mu$ defined by

$$(3.7) \qquad u(t) = \int_0^t (t-s)^\mu v(s)ds, \quad t \geq 0$$

where $v \in \mathscr{U}_0(t_0=0)$. For such u we have

$$(3.8) \qquad u^{(k)}(t) = (\mu)_k \int_0^t (t-s)^{\mu-k} v(s)ds, \quad k = 0,1,\ldots,\mu$$

where $(\mu)_k = \mu!/(\mu-k)!$ so that (3.5) holds.

For a solution of the equivalent system (2.18_i), $i = 0,1$, we then necessarily have $\dot{x}_1(0) = x_1(0) = 0$ by (2.28). When $x(0) = 0$, then $x_0(0) = Ux(0) = 0$ so by (2.29) and (3.7) we have

$$x_0(t) = \int_0^t K_0(t-s)\Gamma_0 v(s)ds, \quad t \geq 0 \tag{3.9}$$

where

$$K_0(t) = \int_0^t e^{\beta(t-\sigma)}\sigma^{\mu}d\sigma \quad , \quad t \geq 0. \tag{3.10}$$

Observe that then $\dot{x}_0(0) = 0$. Hence for controls as in (3.7) we have (3.6).

Substituting (3.8) into (2.28), we may write

$$x_1(t) = \int_0^t K_1(t-s)\Gamma_1 v(s)ds, \quad t \geq 0 \tag{3.11}$$

where

$$K_1(t) = \sum_{k=0}^{\mu-1} (\mu)_k t^{\mu-k}\alpha^k. \tag{3.12}$$

Combining (3.9) and (3.11) and using (2.19), we have

$$x(t) = T\int_0^t K(t-s)\Gamma v(s)ds \tag{3.13}$$

with

$$(3.14) \qquad K(t) = \begin{bmatrix} K_0(t) & 0 \\ 0 & K_1(t) \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix}.$$

We now define a $n \times n$ matrix function W by

$$(3.15) \qquad W(\tau) = \int_0^\tau K(\tau-s)\Gamma\Gamma^T K^T(\tau-s)ds.$$

It is evident that $W(\tau)$ is real symmetric and positive definite or semi-definite. Moreover, $W(\tau)$ is singular for $\tau > 0$ if and only if there is some $c \in \mathbb{R}^n$, $c \neq 0$, such that

$$(3.16) \qquad c^T K(t)\Gamma \equiv 0, \quad 0 \leq t \leq \tau.$$

If $W(\tau)$ is non-singular for some $\tau > 0$, then for any $\zeta \in \mathbb{R}^n$ we can define v by

$$(3.17) \qquad v(t) = \Gamma^T K^T(\tau-t)W^{-1}(\tau)T^{-1}\zeta.$$

With this v in (3.7) the resulting control $u \in \mathscr{U}_\mu$, satisfies (3.5) and, by virtue of (3.13), gives $x(\tau) = \zeta$ when $x(0) = 0$ as well as $\dot{x}(0) = 0$. The theorem will be proved then when we show that (3.3) implies that $W(\tau)$ is non-singular for all $\tau > 0$.

This is included in the following lemma.

Lemma 3.4.

For any $\tau > 0$, $W(\tau)$ defined in (3.15) is non-singular if and only if (3.3) holds.

Proof:

Let $\tau > 0$ be given and let $c \in \mathbb{R}^n$ satisfy $c^T W(\tau) = 0$. This is equivalent to (3.16) which, in turn, is equivalent to

$$c^T K^{(k)}(0)\Gamma = 0 \quad , \quad k = 0,1,2,\ldots \tag{3.18}$$

inasmuch as $K(t)$ is an entire function. From (3.10) one can compute that

$$K_0^{(k)}(0) = \begin{cases} 0, & k = 0,1,\ldots,\mu \\ \mu!\beta^{k-\mu-1}, & k \geq \mu + 1. \end{cases}$$

From (3.12) we get

$$K_1^{(k)}(0) = \begin{cases} \mu!\alpha^{\mu-k}, & k = 1,\ldots,\mu \\ 0, & k = 0 \quad \text{or} \quad k \geq \mu + 1. \end{cases}$$

Accordingly, if we take $c^T = [c_0^T, c_1^T]$ where $c_0 \in \mathbb{R}^r$, $c_1 \in \mathbb{R}^\rho$, then (3.18) is equivalent to

$$c_0^T \beta^k \Gamma_0 = 0 \quad , \quad k = 0,1,2,\ldots \tag{3.19}$$
$$c_1^T \alpha^k \Gamma_1 = 0 \quad , \quad k = 0,1,\ldots,\mu\text{-}1$$

These in turn are equivalent to

$$c_0^T H_0 = 0 \quad , \quad c_1^T H_1 = 0 \tag{3.20}$$

where H_0 and H_1 are defined in (3.4). If $c^T W(\tau) = 0$ and $c \neq 0$, then (3.20) holds with $c_0 \neq 0$ or $c_1 \neq 0$ so (3.3) fails. Conversely, if (3.3) fails, then (3.20) holds for some c_0, c_1 not both zero. The corresponding $c \neq 0$ and satisfies $c^T W(\tau) = 0$. This proves the lemma and completes the proof of Theorem 3.3. $\boxed{\#}$

Lemma 3.5.

Conditions (1.5), (3.3) and

(3.21) $c \in \mathbb{R}^n$, $c^T(-As+B)^{-1}C \equiv 0$, $s \in \mathbb{R}$, implies $c = 0$,

are all equivalent.

Proof:

The equivalence of (1.5) and (3.21) is obvious since $-s$ may be substituted for s in either. To show that (1.5) and (3.3)

are equivalent we use (2.24) to note that

$$(3.22) \qquad S(As+B)T = \mathrm{diag}(sI_r-\beta, I_\rho - s\alpha).$$

It follows that

$$(3.23) \qquad c^T(As+B)^{-1}C = c_0^T(sI_r-\beta)^{-1}\Gamma_0 + c_1^T(I_\rho-s\alpha)^{-1}\Gamma_1$$

where

$$(3.24) \qquad c_0^T = c^TX, \quad c_1^T = c^TY, \quad c^T = [c_0^T, c_1^T]T^{-1}.$$

But for large $|s|$, $s \in \mathbb{R}$, we have

$$(sI_r-\beta)^{-1} = \sum_{k=0}^{\infty} s^{-k-1}\beta^k$$

while for any $s \in \mathbb{R}$ we have

$$(I_\rho - s\alpha)^{-1} = \sum_{k=0}^{\mu-1} s^k\alpha^k$$

by virtue of (2.15). From (3.23) we see then that $c^T(As+B)^{-1}C \equiv 0$, $s \in \mathbb{R}$, is equivalent to (3.19) and hence to (3.20). By virtue of (3.24) it is evident then that (1.5) holds if and only if (3.3) holds. This completes the proof of the lemma. #

We are now in a position to prove our main result:

Proof of Theorem 1.3:

As was pointed out earlier we may assume A is singular and (2.8) holds. Hence if (1.1) is regular and controllable, (3.3) must hold by Theorem 3.2 and this implies (1.5) by Lemma 3.5. Now suppose (1.5) holds. Then (3.3) holds so by Theorem 3.3 system (1.1) is controllable from zero with any $\tau > 0$ and with $u \in \mathcal{U}_\mu$ satisfying (3.5) and the resulting solution x satisfying (3.6). But (1.1) also implies (3.21) by Lemma 3.5 so, by the same argument just given, the system

$$-A\dot{x} + Bx = Cu \tag{3.25}$$

is controllable from zero with any $\tau > 0$ and analogous $u \in \mathcal{U}_\mu$ and solution x. Hence, if (1.1) holds and $\xi, \zeta \in \mathbb{R}^n$, $\tau > 0$ are given, there is a control $u^1 \in \mathcal{U}_\mu$ steering the corresponding solution x^1 of (3.25) from $x^1(0) = 0$ to $x^1(\tau/2) = \xi$ and there is a control $u^2 \in \mathcal{U}_\mu$ steering the corresponding solution x^2 of (1.1) from $x^2(0) = 0$ to $x^2(\tau/2) = \zeta$. The controls u^1 and u^2 satisfy (3.5) and by (3.6) we have $\dot{x}^1(0) = 0$, $\dot{x}^2(0) = 0$. If we define

$$u(t) = \begin{cases} u^1(\tau/2 - t), & 0 \le t \le \tau/2 \\ u^2(t - \tau/2), & t \ge \tau/2, \end{cases} \tag{3.26}$$

then $u \in \mathscr{U}_\mu$. The function x defined by

$$(3.27) \qquad x(t) = \begin{cases} x^1(\tau/2-t) & , \quad 0 \le t \le \tau/2 \\ x^2(t-\tau/2) & , \quad t \ge \tau/2 \end{cases}$$

is continuous and differentiable for $t \ge 0$ (even at $t = \tau/2$) and is a solution of (1.1) with u given by (3.26) which satisfies $x(0) = \xi$ and $x(\tau) = \zeta$. This completes the proof. #

4. Stabilizability of the system.

When (1.1) is regular, the solutions of the homogeneous system

$$(4.1) \qquad A\dot{x} + Bx = 0$$

can be found by setting $u(t) \equiv 0$ in (2.28) and (2.29). One gets $x_0(t) = e^{\beta t}x_0(0)$, $x_1(t) \equiv 0$, and by (2.19), we find the solutions of (4.1) in the form

$$(4.2) \qquad x(t) = Xe^{\beta t}\xi_0 \quad , \quad \xi_0 \in \mathbb{R}^r.$$

Thus the initial values are restricted to the range of P_0 by

$$(4.3) \qquad x(0) = X\xi_0$$

inasmuch as the columns of X are a basis for the range of P_0.

By (2.15) we have $\det(I_\rho - s\alpha) \equiv 1$ so from (3.22) we see that the eigenvalues of β are precisely those λ for which $\Delta(\lambda) = 0$ (cf. (1.2)). The polynomial Δ is of degree r and the solutions (4.2) form an r-dimensional space over $\mathbb{R}$. If all eigenvalues λ have negative real parts, then the zero solution of (4.1) is asymptotically stable relative to the allowable initial values $x(0)$ given in (4.3). However, a bounded input u in the system (1.1) need not result in a bounded response x. Indeed, if $r < n$, $\mu \geq 2$ and $u \in \mathscr{U}_\mu$, we see from (2.28) that then x_1 and hence x may be unbounded when $u^{(\mu-1)}$ is unbounded.

In regard to (1.1) we ask then; what conditions assure that the system can be stabilized by a linear feedback control so that the zero solution of the combined system plus feedback is asymptotically stable relative to arbitrary initial conditions in the state space $\mathbb{R}^n$? It should be clear that when $r < n$, a feedback $u = Gx$ with G being $n \times m$ will not suffice. The combined system is then $A\dot{x} + (B-CG)x = 0$ and the degree of $\det(As+B-CG)$ will still be less than n since A is singular. The corresponding solutions and allowable initial conditions will be constrained to lie in a proper subspace of $\mathbb{R}^n$. To achieve stabilization when A is singular the feedback must contain a term involving $\dot{x}$. Thus we ask for conditions which assure stabilizability in the sense of Definition 1.6. Our principal result in this direction is the following theorem.

Theorem 4.1.

Suppose the system (1.1) is regular and that C is $n \times 1$ $(m = 1)$. If (1.1) is controllable, then it is stabilizable.

Proof:

As before, we treat (1.1) in the decomposed form (2.18_i), $i = 0,1$, under the assumption (2.8); the extreme cases $r = 0$ and $r = n$ (A non-singular) are included thereby when vacuous terms are omitted. By hypothesis (1.1) is controllable, so (1.5) holds by Theorem 1.3 and hence (3.3) holds by Lemma 3.5. Since Γ_0 is $r \times 1$ $(m = 1)$, then H_0 is $r \times r$ and non-singular. It follows (Theorem 7, p.90, [5]) that there exists a non-singular $r \times r$ matrix F_0 such that if

$$y_0 = F_0 x_0, \tag{4.4}$$

then (2.18_0) is equivalent to

$$\dot{y}_0 - M(\beta) y_0 = e_r u \tag{4.5}$$

where e_r is the r^{th} column of I_r and $M(\beta)$ is a companion matrix for the characteristic polynomial of β; specifically,

$$(4.6)\qquad M(\beta) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{r-1} \end{bmatrix}$$

where $\det(\lambda I_r - \beta) = \lambda^r + b_{r-1}\lambda^{r-1} + \cdots + b_0$.

The matrix H_1 in (3.4) is $\rho \times \mu$ so we must have $\mu \geq \rho$ since rank $H_1 = \rho$. Since α is $\rho \times \rho$ and satisfies (2.15) we must have $\mu \leq \rho$. Hence $\mu = \rho$ and H_1 is square and non singular. There is then a non-singular $\rho \times \rho$ matrix F_1 such that if

$$(4.7)\qquad y_1 = F_1 x_1,$$

then (2.18_1) is equivalent to

$$(4.8)\qquad -M(\alpha)\dot{y}_1 + y_1 = e_\rho u$$

where e_ρ is the ρ^{th} column of I_ρ and $M(\alpha)$ a companion matrix for the characteristic polynomial of α; specifically, $M(\alpha)$ is of the same form as $M(\beta)$ in (4.6) except that the last row is zero. Letting

$$y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \tag{4.9}$$

and combining (4.5) and (4.8), we have the system

$$\begin{bmatrix} I_r & 0 \\ 0 & -M(\alpha) \end{bmatrix} y + \begin{bmatrix} -M(\beta) & 0 \\ 0 & I_\rho \end{bmatrix} y = \begin{bmatrix} e_r \\ e_\rho \end{bmatrix} u \tag{4.10}$$

which is equivalent to (1.1) under the hypotheses of the theorem.

It is convenient now to replace (4.10) by the system equivalent to it obtained by subtracting the last equation of the system from the r^{th} equation. Since the last row of $M(\alpha)$ is zero, this is

$$\begin{bmatrix} I_r & 0 \\ 0 & -M(\alpha) \end{bmatrix} \dot{y} + \begin{bmatrix} -M(\beta) & -D_0 \\ 0 & I_\rho \end{bmatrix} y = \begin{bmatrix} 0 \\ e_\rho \end{bmatrix} u \tag{4.11}$$

where D_0 is $r \times \rho$ and all its elements are zero except for a one in the r^{th} row and ρ^{th} column.

Next let $e_1^T = [1,0,\ldots,0]$ be $1 \times \rho$ and replace u in (4.11) by

$$u = e_1^T \dot{y}_1 - v. \tag{4.12}$$

The result may be written, after changing signs in the last ρ

equations of the system, in the form

$$(4.13)\quad \begin{bmatrix} I_r & 0 \\ 0 & M(\alpha) + e_\rho e_1^T \end{bmatrix} \dot{y} - \begin{bmatrix} M(\beta) & D_0 \\ 0 & I_\rho \end{bmatrix} y = \begin{bmatrix} 0 \\ e_\rho \end{bmatrix} v.$$

We now define two permutation matrices π_0 and π_1: π_0 is $\rho \times \rho$ and reverses the order of the columns of a matrix (of ρ columns) when used as a postfactor; π_1 is $\rho \times \rho$ and reverses the order of the first $\rho - 1$ rows and leaves the last row unaffected when used as a prefactor on a matrix with ρ rows. Note that $\pi_0^2 = \pi_1^2 = I_\rho$. In (4.13) we make the substitution

$$(4.14)\quad y = [\mathrm{diag}(I_r, \pi_0)]\, z$$

and multiply the result by the matrix $\mathrm{diag}(I_r, \pi_1)$. Since $\pi_1 e_\rho = e_\rho$ and $\pi_1(M(\alpha)+e_\rho e_1^T) = I_\rho$, the resulting equivalent system takes the form

$$(4.15)\quad \dot{z} = \mathscr{A} z + bv$$

where

$$(4.16)\quad \mathscr{A} = \begin{bmatrix} M(\beta) & D_0\pi_0 \\ 0 & \pi_1\pi_0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n$$

Note that all elements of $D_0\pi_0$ are zero except for a one in the r^{th} row and first column. Also, the element in the i^{th} row and $i+1^{th}$ column of $\pi_1\pi_0$, $i = 1,\ldots,\rho-1$, is a one and all others are zeros except for a one in the ρ^{th} row and first column. Hence, from the form of $M(\beta)$ in (4.6), we see that $\mathcal{A}$ has ones just above the main diagonal and all zeroes everywhere above these. It is then clear from the form of b in (4.16) that

$$H = [b, \mathcal{A}b, \ldots, \mathcal{A}^{n-1}b] \tag{4.17}$$

is non-singular. Hence there exists a feedback (Theorem 9, p.97, [5])

$$v = g^T z \quad , \quad g \in \mathbb{R}^n \tag{4.18}$$

which when substituted into (4.15) gives the system

$$\dot{z} = (\mathcal{A} + bg^T)z \tag{4.19}$$

with $\mathcal{A} + bg^T$ stable; that is, g can be chosen so that all eigenvalues of $\mathcal{A} + bg^T$ have negative real parts and all solutions of (4.19) tend to zero exponentially as $t \to +\infty$.

The substitutions (4.4) and (4.7) can be written as

$$y = [\mathrm{diag}(F_0, F_1)]T^{-1}x \tag{4.20}$$

and (4.14) can be written

$$z = [\mathrm{diag}(I_r, \pi_0)]y \tag{4.21}$$

since $\pi_0^2 = I_\rho$. The combined feedback resulting from (4.12) and (4.18) is thus of the form

$$u = e_1^T\dot{y}_1 - g^T z = g_0^T x + g_1^T \dot{x} \tag{4.22}$$

for some $g_0, g_1 \in \mathbb{R}^n$; specifically,

$$g_0^T = -g^T \begin{bmatrix} F_0 U \\ \pi_0 F_1 V \end{bmatrix} \quad ; \quad g_1^T = e_1^T F_1 V. \tag{4.23}$$

Using (4.20) and (4.21), we can express (4.19) in terms of x. The resulting equation, when multiplied by the inverses (in reverse order) of the several matrices used, in effect, as prefactors in going from (1.1) to (4.19), will produce the form (1.7) with $G_0 = g_0^T$, $G_1 = g_1^T$; that is, (1.7) with

$$\tilde{A} = A - Cg_1^T \quad , \quad \tilde{B} = B - Cg_0^T. \tag{4.24}$$

The term $\tilde{A}\dot{x}$ arises from the term $\dot{z} = I_n\dot{z}$ in (4.19) so $\tilde{A}$ must be non-singular and all solutions $x(t)$ of (1.7) tend to zero

exponentially as $t \to \infty$ since all solutions $z(t)$ of (4.19) do so. This completes the proof of the theorem. $\boxed{\#}$

In the above, C was assumed to be $n \times 1$. We can use this to prove the stabilizability of (1.1) for some cases when C is $n \times m$ with $m > 1$.

Corollary 4.2.

Suppose the system (1.1) is regular, that rank $A = n - 1$ and that the zeros of $\Delta(s) = \det(As+B)$ are distinct and none is zero. If (1.1) is controllable, then it is stabilizable.

Proof:

Since rank $A = n - 1$, A is singular so the polynomial $\Delta(s)$ has degree $r < n$. ($\Delta(s) \not\equiv 0$ since (1.1) is regular.) We may write

$$(4.25) \qquad \Delta(s) = d_0 s^r + \cdots + d_r, \quad d_0 \neq 0, \quad d_r \neq 0,$$

the condition $d_r \neq 0$ being a result of the hypotehsis that $\Delta(0) \neq 0$. From (4.25) we find $(z \in \mathbb{C})$

$$\det(A+zB) = z^n \Delta(1/z) = z^{n-r}(d_r z^r + \ldots + d_0).$$

The cofactor of at least one element in $A + zB$ is non-zero at $z = 0$ since rank $A = n - 1$. Hence since $d_0 \neq 0$

$$\lim_{z\to 0} z^{n-r}(A+zB)^{-1}$$

exists and is non-zero so $(A+zB)^{-1}$ has a pole of order $n - r$ at $z = 0$. Thus μ in (2.1) satisfies $\mu = n - r = \rho$. Since $r < n$, we have $\rho > 0$ and, again for notational convenience only, we assume $r > 0$ so that (2.8) holds.

Now let

$$E = \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix} \tag{4.26}$$

where β and α are defined in (2.14) as in the development in §2. Consider

$$H = [\Gamma, E\Gamma, \ldots, E^{n-1}\Gamma] \tag{4.27}$$

where Γ is as in (3.14). Since $\mu = \rho$ and $\alpha^{\mu} = 0$, we find

$$H = \begin{bmatrix} \hat{H}_0 & \beta^{\rho}H_0 \\ H_1 & 0 \end{bmatrix}$$

where $\hat{H}_0 = [\Gamma_0, \beta\Gamma_0, \ldots, \beta^{\rho-1}\Gamma_0]$ and H_0 and H_1 are as in (3.4). But the eigenvalues of β are the zeros of $\Delta(s)$ so our hypotehses imply that β is non-singular. The controllability

of (1.1) implies (3.3) by virtue of Theorem 1.3 and Lemma 3.5. It follows then that rank $\tilde{H} = n$.

Since $\mu = \rho$ and α is $\rho \times \rho$ and nilpotent of order μ, the Jordan canonical form for α is $M(\alpha)$ as described just after (4.8). Any Jordan form for β is matrix diagonal with distinct diagonal elements and none zero by virtue of our hypothesis regarding the zeros of $\Delta(s)$. Hence in any Jordan form for E no two Jordan blocks have the same eigenvalue. It follows then from Theorem 6, p. 86 of [5] that there exists a $c \in \mathbb{R}^m$ such that

$$\tilde{H} = [\tilde{\Gamma}, E\tilde{\Gamma}, \ldots, E^{n-1}\tilde{\Gamma}], \qquad \tilde{\Gamma} = \Gamma c$$

has rank n. This in turn implies

$$\text{rank } \tilde{H}_0 = r, \qquad \text{rank } \tilde{H}_1 = \rho \tag{4.28}$$

where $\tilde{H}_0$ and $\tilde{H}_1$ are as in (3.4) with Γ_0 and Γ_1 replaced by $\Gamma_0 c$ and $\Gamma_1 c$, respectively.

Again using Lemma 3.5 and Theorem 1.3, we may now conclude that

$$A\dot{x} + Bx = \tilde{C}\tilde{u} \quad , \quad \tilde{C} = Cc, \quad \tilde{u} \in \mathbb{R}^1 \tag{4.29}$$

is controllable. Since now $\tilde{C}$ is $n \times 1$, Theorem 4.1 implies

that (4.29) is stabilizable. But (4.29) is (1.1) with $u = c\tilde{u}$ and the stabilizing feedback $\tilde{u} = g_0^T x + g_1^T \dot{x}$ for (4.29) determines a stabilizing feedback (1.6) for (1.1) with $G_0 = cg_0^T$, $G_1 = cg_1^T$. #

It may be noted that the hypotheses in Corollary 4.2 do not imply $r = n - 1$. The following is a case in which $r < n - 1$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Here $\operatorname{rank} A = 2 = n - 1$ but $\Delta(s) = s + 1$ has degree $1 < n - 1$.

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A concept of controllability for systems $A\dot{x} + Bx$ $Cu(t)$ in which A may be singular is introduced. When $\det(As+B) \neq 0$, s C, this is shown to be equivalent to the condition that $C^T(As+B)^{-1}C = 0$ implies c 0. It is also shown that when such a system is controllable and C is a column vector, then there exists a feedback $u = g_0^T x + g_1^T \dot{x}$ such that $A - Cg_1^T$ is non singular and all solutions of $A\dot{x} + Bx = C(g_0^T x + g_1^T)$ decay exponentially.

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